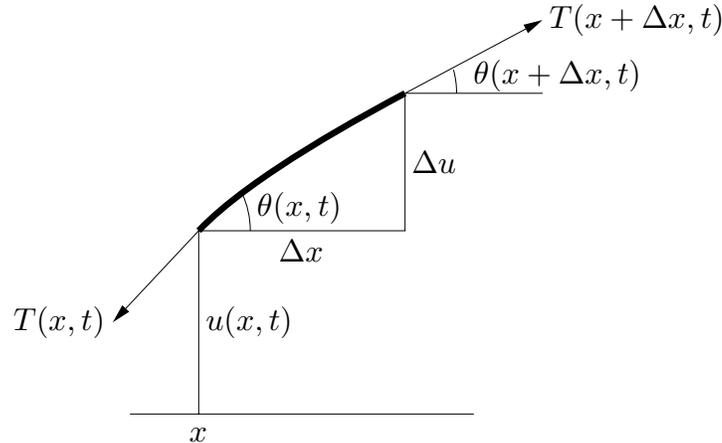


Derivation of the Wave Equation

In these notes we apply Newton's law to an elastic string, concluding that small amplitude transverse vibrations of the string obey the wave equation. Consider a tiny element of the string.



The basic notation is

$u(x, t)$ = vertical displacement of the string from the x axis at position x and time t

$\theta(x, t)$ = angle between the string and a horizontal line at position x and time t

$T(x, t)$ = tension in the string at position x and time t

$\rho(x)$ = mass density of the string at position x

The forces acting on the tiny element of string are

- tension pulling to the right, which has magnitude $T(x + \Delta x, t)$ and acts at an angle $\theta(x + \Delta x, t)$ above horizontal
- tension pulling to the left, which has magnitude $T(x, t)$ and acts at an angle $\theta(x, t)$ below horizontal and, possibly,
- various external forces, like gravity. We shall assume that all of the external forces act vertically and we shall denote by $F(x, t)\Delta x$ the net magnitude of the external force acting on the element of string.

The mass of the element of string is essentially $\rho(x)\sqrt{\Delta x^2 + \Delta u^2}$ so the vertical component of Newton's law says that

$$\rho(x)\sqrt{\Delta x^2 + \Delta u^2}\frac{\partial^2 u}{\partial t^2}(x, t) = T(x + \Delta x, t) \sin \theta(x + \Delta x, t) - T(x, t) \sin \theta(x, t) + F(x, t)\Delta x$$

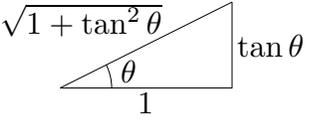
Dividing by Δx and taking the limit as $\Delta x \rightarrow 0$ gives

$$\begin{aligned} \rho(x) \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} \frac{\partial^2 u}{\partial t^2}(x, t) &= \frac{\partial}{\partial x} [T(x, t) \sin \theta(x, t)] + F(x, t) \\ &= \frac{\partial T}{\partial x}(x, t) \sin \theta(x, t) + T(x, t) \cos \theta(x, t) \frac{\partial \theta}{\partial x}(x, t) + F(x, t) \end{aligned} \quad (1)$$

We can dispose of all the θ 's by observing from the figure that

$$\tan \theta(x, t) = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}(x, t)$$

which implies, using the figure on the right below, that

$$\begin{aligned} \sin \theta(x, t) &= \frac{\frac{\partial u}{\partial x}(x, t)}{\sqrt{1 + \left(\frac{\partial u}{\partial x}(x, t)\right)^2}} & \cos \theta(x, t) &= \frac{1}{\sqrt{1 + \left(\frac{\partial u}{\partial x}(x, t)\right)^2}} \\ \theta(x, t) &= \tan^{-1} \frac{\partial u}{\partial x}(x, t) & \frac{\partial \theta}{\partial x}(x, t) &= \frac{\frac{\partial^2 u}{\partial x^2}(x, t)}{1 + \left(\frac{\partial u}{\partial x}(x, t)\right)^2} \end{aligned}$$


Substituting these formulae into (1) give a horrendous mess. However, we can get considerable simplification by looking only at small vibrations. By a small vibration, we mean that $|\theta(x, t)| \ll 1$ for all x and t . This implies that $|\tan \theta(x, t)| \ll 1$, hence that $\left|\frac{\partial u}{\partial x}(x, t)\right| \ll 1$ and hence that

$$\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} \approx 1 \quad \sin \theta(x, t) \approx \frac{\partial u}{\partial x}(x, t) \quad \cos \theta(x, t) \approx 1 \quad \frac{\partial \theta}{\partial x}(x, t) \approx \frac{\partial^2 u}{\partial x^2}(x, t) \quad (2)$$

Substituting these into equation (1) give

$$\rho(x) \frac{\partial^2 u}{\partial t^2}(x, t) = \frac{\partial T}{\partial x}(x, t) \frac{\partial u}{\partial x}(x, t) + T(x, t) \frac{\partial^2 u}{\partial x^2}(x, t) + F(x, t) \quad (3)$$

which is indeed relatively simple, but still exhibits a problem. This is one equation in the two unknowns u and T .

Fortunately there is a second equation lurking in the background, that we haven't used. Namely, the horizontal component of Newton's law of motion. As a second simplification, we assume that there are only transverse vibrations. Our tiny string element moves only vertically. Then the net horizontal force on it must be zero. That is,

$$T(x + \Delta x, t) \cos \theta(x + \Delta x, t) - T(x, t) \cos \theta(x, t) = 0$$

Dividing by Δx and taking the limit as Δx tends to zero gives

$$\frac{\partial}{\partial x} [T(x, t) \cos \theta(x, t)] = 0$$

For small amplitude vibrations, $\cos \theta$ is very close to one and $\frac{\partial T}{\partial x}(x, t)$ is very close to zero. In other words T is a function of t only, which is determined by how hard you are pulling on the ends of the string at time t . So for small, transverse vibrations, (3) simplifies further to

$$\rho(x) \frac{\partial^2 u}{\partial t^2}(x, t) = T(t) \frac{\partial^2 u}{\partial x^2}(x, t) + F(x, t) \quad (4)$$

In the event that the string density ρ is a constant, independent of x , the string tension $T(t)$ is a constant independent of t (in other words you are not continually playing with the tuning pegs) and there are no external forces F we end up with

$$\boxed{\frac{\partial^2 u}{\partial t^2}(x, t) = c^2 \frac{\partial^2 u}{\partial x^2}(x, t)}$$

where

$$c = \sqrt{\frac{T}{\rho}}$$